Important results and their derivations | A

A.1 Electrostatic Dipole : Potential and Field Derivation

Scalar potential of a dipole

Consider two point charges $+q$ and $-q$ separated by a small vector $\mathbf{d}$, with the positive charge located at $\mathbf{r}_+ = +\frac{1}{2}\mathbf{d}$ and the negative at $\mathbf{r}_- = -\frac{1}{2}\mathbf{d}$. The potential at field point $\mathbf{r} = r\mathbf{u_r}$ is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{r}_+|} - \frac{q}{|\mathbf{r}-\mathbf{r}_-|}\right). \tag{A.1}$$

We define the dipole moment

$$\mathbf{p} = q\,\mathbf{d}. \tag{A.2}$$

Assuming that the observation point is far from the dipole ($r \gg d$), we perform a Taylor expansion of each term to first order in $\mathbf{d}$:

$$\frac{1}{|\mathbf{r} \mp \frac{1}{2}\mathbf{d}|} \simeq \frac{1}{r} \pm \frac{1}{2}\,\mathbf{d}\cdot\nabla\left(\frac{1}{r}\right) + \mathcal{O}(d^2). \tag{A.3}$$

Hence,

$$\frac{1}{|\mathbf{r} - \frac{1}{2}\mathbf{d}|} - \frac{1}{|\mathbf{r} + \frac{1}{2}\mathbf{d}|} \simeq -\mathbf{d}\cdot\nabla\left(\frac{1}{r}\right). \tag{A.4}$$

Substituting into the potential :

$$\Phi(\mathbf{r}) = -\frac{q}{4\pi\varepsilon_0}\,\mathbf{d}\cdot\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{p}}{4\pi\varepsilon_0}\cdot\nabla\left(\frac{1}{r}\right). \tag{A.5}$$

Recall that iun spherical coordinates and for a scalar quantity f :

$$\mathbf{grad}(f) = \nabla f = \frac{\partial f}{\partial r}\mathbf{u_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{u}_\phi$$

$\nabla(1/r) = -\mathbf{r}/r^3$,

Hence,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{u_r}}{r^2}, \qquad r \neq 0. \tag{A.6}$$

This is the potential of an electrostatic dipole : the two charges, positive and negative do not entirely cancel out, but the potential rapidly vanishes away from the two charges.

Alternative derivation (multipole expansion)

We can retrieve the exact same previous result for an arbitrary charge distribution, by expanding the general expression of the scalar potential. For a localized charge distribution $\rho(\mathbf{r}')$,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r'.$$

Expanding $1/|\mathbf{r}-\mathbf{r}'|$ for $r \gg r'$:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \mathbf{r}'\cdot\nabla\left(\frac{1}{r}\right) + \cdots$$

and defining $\mathbf{p} = \int \rho(\mathbf{r}')\,\mathbf{r}'\, d^3r'$, the monopole term (total charge) vanishes for a neutral system, therefore the lowest order term, corresponding to the dipole moment, is the preponderant term on the scalar potential.

$$\Phi(\mathbf{r}) = \mathbf{p}\cdot\nabla\left(\frac{1}{4\pi\varepsilon_0\, r}\right) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}. \tag{A.7}$$

In what follows, we will refer to the three spatial cartesian coordinates of the system using an index, such as x_i with $i = 1{,}2{,}3$, forming the coordinate systems so that the field position reads :

$$\mathbf{r} = x_1\mathbf{u_{x_1}} + x_2\mathbf{u_{x_2}} + x_3\mathbf{u_{x_3}}$$

And note that $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

A compact way to write the expressions of the potentials without fully expanding the dot product in the cartesian basis is to use the so-called Einstein notations :

$$\Phi = p_j x_j/4\pi\varepsilon_0 r^3 \quad \text{implicitly means} \quad \Phi = \sum_{j=1,2,3} p_j x_j/4\pi\varepsilon_0 r^3$$

This is often referred as a "summation over repeated indices". The spatial derivatives $\frac{\partial}{\partial x_i}$ are written as ∂_i.

Electric field of an electrostatic dipole

The electric field is the negative gradient of the potential :

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\nabla\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right).$$

The rest of the calculation is a careful calculation of the derivatives, decomposed over the different vector components. The calculation is easier when performing the spatial derivatives in cartesian coordinates. The full expansion reads :

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\nabla\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right) = -\frac{1}{4\pi\varepsilon_0}\begin{pmatrix}\frac{\partial}{\partial x_1}(p_1x_1+p_2x_2+p_3x_3)r^{-3}\\ \frac{\partial}{\partial x_2}(p_1x_1+p_2x_2+p_3x_3)r^{-3}\\ \frac{\partial}{\partial x_3}(p_1x_1+p_2x_2+p_3x_3)r^{-3}\end{pmatrix}$$

As an example, we provide the detailed calculation of a single component of the vector field.

We first compute a single component of the electric field, using the same label i that refers to the coordinates.

$$E_i = -\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial x_1}(p_1x_1+p_2x_2+p_3x_3)r^{-3} \tag{A.8}$$

$$= -\frac{1}{4\pi\varepsilon_0}\left(p_1r^{-3}+p_2x_2\frac{\partial}{\partial x_1}r^{-3}+p_3x_3\frac{\partial}{\partial x_1}r^{-3}\right) \tag{A.9}$$

The spatial derivatives of the distance r are easily computed :

Spatial derivatives

$$\frac{\partial r^k}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1^2+x_2^2+x_3^2)^{k/2} = 2x_1\cdot\frac{k}{2}(x_1^2+x_2^2+x_3^2)^{k/2-1} = kx_1r^{k-1} \tag{A.10}$$

Hence,

$$E_i = -\frac{1}{4\pi\varepsilon_0}\left(p_1r^{-3}+p_2x_2\frac{\partial}{\partial x_1}r^{-3}++p_3x_3\frac{\partial}{\partial x_1}r^{-3}\right) \tag{A.11}$$

$$= -\frac{1}{4\pi\varepsilon_0}\left(p_1r^{-3}-3p_2x_2x_1r^{-5}-3p_3x_3x_1r^{-5}\right) \tag{A.12}$$

This whole calculation can be written using the Einstein's notations, that we will now constantly use throughout the rest of the chapter : Writing in components ($\Phi = p_j x_j / 4\pi\varepsilon_0 r^3$) :

$$\begin{aligned} E_i &= -\frac{1}{4\pi\varepsilon_0} p_j \partial_i (x_j r^{-3}) \\ &= -\frac{1}{4\pi\varepsilon_0} p_j \left(\delta_{ij} r^{-3} - 3 x_i x_j r^{-5}\right) \\ &= \frac{1}{4\pi\varepsilon_0} \left(3\frac{x_i x_j p_j}{r^5} - \frac{p_i}{r^3}\right). \end{aligned} \tag{A.13}$$

In vector form,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left[\frac{3\,\mathbf{r}\,(\mathbf{r}\cdot\mathbf{p})}{r^5} - \frac{\mathbf{p}}{r^3}\right], \qquad r \neq 0. \tag{A.14}$$

Dyadic form

The electric field generated by an electrostatic dipole is not oriented parallel to this dipole but as a spatially varying orientation. This means that the best mathematical way to represent the relation between a dipole **p** and the induced field **E** is via a matrix representing the linear application. The matrix is a tensor or sometimes called a dyadic.

We introduce the tensor (or dyadic) product :

$$\mathbf{rr} = \mathbf{r} \otimes \mathbf{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\ x_2 x_1 & x_2^2 & x_2 x_3 \\ x_3 x_1 & x_3 x_2 & x_3^2 \end{pmatrix}$$

It will be lead us to write the potential in the so-called dyadic form, using a very convenient identity :

$$\mathbf{rr}\cdot\mathbf{p} = \begin{pmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\ x_2 x_1 & x_2^2 & x_2 x_3 \\ x_3 x_1 & x_3 x_2 & x_3^2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} x_1(x_1 p_1 + x_2 p_2 + x_3 p_3) \\ x_2(x_1 p_1 + x_2 p_2 + x_3 p_3) \\ x_3(x_1 p_1 + x_2 p_2 + x_3 p_3) \end{pmatrix} = (x_1 p_1 + x_2 p_2 + x_3 p_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (\mathbf{r}\cdot\mathbf{p})\mathbf{r}$$

We also introduce the identity dyadic **I** (which is basically the identity matrix). Using the previous identity, the field can be compactly written as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{3\,\mathbf{rr}}{r^5} - \frac{\mathbf{I}}{r^3}\right)\cdot\mathbf{p}. \tag{A.15}$$

Equivalently, in terms of the unit vector $\mathbf{u_r} = \mathbf{r}/r$,

Electric field of an electrostatic dipole - Dyadic form

$$\mathbf{E}(\mathbf{r}) = \frac{-1}{4\pi\varepsilon_0}\left(\frac{\mathbf{I} - 3\,\mathbf{u_r}\mathbf{u_r}}{r^3}\right)\cdot\mathbf{p}. \tag{A.16}$$

Remarks

- The potential scales as $1/r^2$ and the field as $1/r^3$.
- On the dipole axis ($\mathbf{p} \parallel \mathbf{u_r}$) : $\mathbf{E} = (1/2\pi\varepsilon_0)(\mathbf{p}/r^3)$. This is where the field is maximum.
- On the equatorial plane ($\mathbf{p} \perp \mathbf{u_r}$) : $\mathbf{E} = -(1/4\pi\varepsilon_0)(\mathbf{p}/r^3)$. This is where the field is minimum.

A.2 Fields of a Harmonic Electric Dipole

Radiation is the consequence of charges being in motion. In optics, talking about dipoles often refers implictly to the case of the oscillating dipole, observed in the far-field : a point electric dipole at the origin but with harmonic oscillation :

$$\mathbf{p}(t) = \mathbf{p}_0 e^{-i\omega t}, \quad k = \frac{\omega}{c}, \quad \mathbf{u_r} = \frac{\mathbf{r}}{r}.$$

where $\mathbf{p_0}$ is the complex amplitude of the oscillating dipole. However, not all fields produced by this oscillating dipole are radiated fields. We perform here a complete derivation of the total field.

Vector potential (Lorenz gauge)

The expression of the potential generated by a distribution of source is a fundamental result of electromagnetism that can be derived from the Maxwell's equation with a source term, and the Green's function method. We will not provide here the full derivation of this specific problem. When considering an oscillating point dipole, the corresponding retarded vector potential is

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_r)}{r} = \frac{\mu_0}{4\pi}\frac{-i\omega\mathbf{p}_0 e^{ikr}e^{-i\omega t}}{r}.$$

We now move to the harmonic regime at ω and consider the complex amplitudes of the different quantities : we therefore get rid of the time varying exponential in the upcoming expressions.

Magnetic field

The magnetic field derives directly from the definition of the vector potential :

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left[\frac{-i\omega\mu_0}{4\pi} \frac{e^{ikr}}{r} \mathbf{p}_0 \right] = \frac{-i\omega\mu_0}{4\pi} \nabla \left(\frac{e^{ikr}}{r} \right) \times \mathbf{p}_0.$$

The gradient of the scalar factor is radial, and we use once again the spherical coordinates for simplicity :

$$\nabla \frac{e^{ikr}}{r} = \mathbf{u_r} \left(\frac{\partial}{\partial r} \frac{e^{ikr}}{r} \right) = \mathbf{u_r} \left(\frac{ike^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right).$$

Thus the magnetic field is

Magnetic field of an oscillating dipole $\mathbf{p_0}$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 \omega}{4\pi} e^{ikr} \left(\frac{k}{r} + \frac{i}{r^2} \right) \mathbf{u_r} \times \mathbf{p}_0 \tag{A.17}$$

Electric field : step-by-step curl

Using Maxwell's equation for time-harmonic fields :

$$\mathbf{E} = \frac{i}{\omega \varepsilon_0 \mu_0} \nabla \times \mathbf{B} = \frac{ic^2}{\omega} \nabla \times \mathbf{B}.$$

We write

$$\mathbf{B} = f(r) \mathbf{u_r} \times \mathbf{p}_0, \quad f(r) = \frac{\mu_0 \omega}{4\pi} e^{ikr} \left(\frac{k}{r} + \frac{i}{r^2} \right).$$

To compute $\mathbf{E}$, we therefore need the compute the curl of $\mathbf{B}$, $\nabla \times \tilde{\mathbf{B}}$.

We have $\tilde{\mathbf{B}} = f(r)\, \mathbf{u_r} \times \mathbf{p}_0$.

We first use a composition rule $\nabla \times (\phi \mathbf{a}) = (\nabla \phi) \times \mathbf{a} + \phi\, \nabla \times \mathbf{a}$ with $\phi = f(r)$ and $\mathbf{a} = \mathbf{u_r} \times \mathbf{p}_0$:

$$\nabla \times \tilde{\mathbf{B}} = (\nabla f(r)) \times (\mathbf{u_r} \times \mathbf{p}_0) + f(r) \nabla \times (\mathbf{u_r} \times \mathbf{p}_0).$$

We now derive the different terms in this expression.

Compute the first term $(\nabla f(r)) \times (\mathbf{u_r} \times \mathbf{p}_0)$

Compute ∇f . Since $f(r) = \frac{\mu_0 \omega}{4\pi} e^{ikr}(k/r + i/r^2)$, we differentiate with respect to r :

$$f'(r) = \frac{\mu_0 \omega}{4\pi} e^{ikr} \left[ik\left(\frac{k}{r} + \frac{i}{r^2}\right) - \frac{k}{r^2} - \frac{2i}{r^3} \right].$$

so

$$f'(r) = \frac{\mu_0 \omega}{4\pi} e^{ikr} \left(\frac{ik^2}{r} - \frac{2k}{r^2} - \frac{2i}{r^3} \right).$$

Because $\nabla f = f'(r)\mathbf{u_r}$, we get the first term :

$$(\nabla f) \times (\mathbf{u_r} \times \mathbf{p}_0) = f'(r)\, \mathbf{u_r} \times (\mathbf{u_r} \times \mathbf{p}_0) \tag{B}$$

And we use a final identity :

$$\mathbf{u_r} \times (\mathbf{u_r} \times \mathbf{p}_0) = \left[\mathbf{u_r}(\mathbf{u_r} \cdot \mathbf{p}_0) - \mathbf{p}_0\right]$$

The double vector product of $\mathbf{p}_0$ with respect to the unit position vector $\mathbf{u_r}$ is equivalent to taking the transverse component of a vector (with a plus or minus sign depending on the orientation of the product.

We have therefore the final result for the first term of the curl of $\mathbf{B}$:

$$(\nabla f(r)) \times (\mathbf{u_r} \times \mathbf{p}_0) = \frac{\mu_0 \omega}{4\pi} e^{ikr} \left(\frac{ik^2}{r} - \frac{2k}{r^2} - \frac{2i}{r^3} \right) \left[\mathbf{u_r}(\mathbf{u_r} \cdot \mathbf{p}_0) - \mathbf{p}_0\right]$$

Compute the second term $f(r)\nabla \times (\mathbf{u_r} \times \mathbf{p}_0)$

We can expand the vector product $\nabla \times \left(\mathbf{u_r} \times \mathbf{p}_0\right)$ using some vector analysis relations :

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}.$$

Take $\mathbf{u} = \mathbf{u_r}$, $\mathbf{v} = \mathbf{p}_0$ (constant with respect to spatial coordinates), then

$$\nabla \times (\mathbf{u_r} \times \mathbf{p}_0) = \mathbf{u_r}(\nabla \cdot \mathbf{p}_0) - \mathbf{p}_0(\nabla \cdot \mathbf{u_r}) + (\mathbf{p}_0 \cdot \nabla)\mathbf{u_r} - (\mathbf{u_r} \cdot \nabla)\mathbf{p}_0.$$

Since $\mathbf{p}_0$ is constant, $\nabla \cdot \mathbf{p}_0 = 0$ and $(\mathbf{u_r} \cdot \nabla)\mathbf{p}_0 = 0.$

Therefore

$$\nabla \times (\mathbf{u_r} \times \mathbf{p}_0) = -\mathbf{p}_0(\nabla \cdot \mathbf{u_r}) + (\mathbf{p}_0 \cdot \nabla)\mathbf{u_r}.$$

Computing $\nabla \cdot \mathbf{u_r}$ Let

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}, \qquad r = \sqrt{x^2 + y^2 + z^2}, \qquad \mathbf{u_r} = \frac{\mathbf{r}}{r}.$$

We aim to compute

$$\nabla \cdot \mathbf{u_r} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right).$$

We differentiate each term. For the x-component :

$$\frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r} + x\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{1}{r} - x\,\frac{1}{r^2}\frac{\partial r}{\partial x} = \frac{1}{r} - x\,\frac{1}{r^2}\frac{x}{r} = \frac{1}{r} - \frac{x^2}{r^3}.$$

By symmetry,

$$\frac{\partial}{\partial y}\left(\frac{y}{r}\right) = \frac{1}{r} - \frac{y^2}{r^3}, \qquad \frac{\partial}{\partial z}\left(\frac{z}{r}\right) = \frac{1}{r} - \frac{z^2}{r^3}.$$

We sum all three components

$$\nabla \cdot \mathbf{u_r} = \left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \left(\frac{1}{r} - \frac{y^2}{r^3}\right) + \left(\frac{1}{r} - \frac{z^2}{r^3}\right).$$

Simplify :

$$\nabla \cdot \mathbf{u_r} = \frac{3}{r} - \frac{x^2 + y^2 + z^2}{r^3} = \frac{3}{r} - \frac{r^2}{r^3} = \frac{2}{r}. \tag{A.18}$$

Compute $(\mathbf{p_0} \cdot \nabla)\mathbf{r}$ We derive the action of the directional derivative $(\mathbf{p}_0 \cdot \nabla)$ on the unit radial vector $\mathbf{u_r}$.

Write components $\hat{r}_i = x_i/r$ with $x_1 = x$, $x_2 = y$, $x_3 = z$. Then

$$\frac{\partial \hat{r}_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{x_i}{r}\right) = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3},$$

as shown by differentiating $1/r$ or by the result used earlier.

Now compute

$$\left((\mathbf{p} \cdot \nabla)\mathbf{u_r}\right)_i = p_j \frac{\partial \hat{r}_i}{\partial x_j} = p_j\left(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\right) = \frac{p_i}{r} - \frac{x_i (p_j x_j)}{r^3}.$$

Recognize $p_j x_j = \mathbf{p} \cdot \mathbf{r}$ and $x_i/r = \hat{r}_i$. Thus

$$\left((\mathbf{p} \cdot \nabla)\mathbf{u_r}\right)_i = \frac{p_i}{r} - \frac{\hat{r}_i (\mathbf{p} \cdot \mathbf{r})}{r^2} = \frac{1}{r}\left(p_i - (\mathbf{p} \cdot \mathbf{u_r})\,\hat{r}_i\right).$$

Returning to vector form,

$$(\mathbf{p}_0 \cdot \nabla)\mathbf{u_r} = \frac{1}{r}\Big(\mathbf{p}_0 - (\mathbf{p}_0 \cdot \mathbf{u_r})\,\mathbf{u_r}\Big).$$

Final result for the second term Combining all previous intermediate steps :

$$\nabla \times (\mathbf{u_r} \times \mathbf{p}_0) = -\frac{2}{r}\mathbf{p}_0 + \frac{1}{r}\Big(\mathbf{p}_0 - (\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r}\Big).$$

We collect terms, not forgetting the first prefactor $f(r)$

$$f(r)\nabla \times (\mathbf{u_r} \times \mathbf{p}_0) = -\frac{f(r)}{r}\big(\mathbf{p}_0 + (\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r}\big) = -\frac{\mu_0\omega}{4\pi}e^{ikr}\big(k/r^2 + i/r^3\big)\big(\mathbf{p}_0 + (\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r}\big).$$

Complete expression

We combine the previous results to get the full curl of **B**

$$\nabla \times \tilde{\mathbf{B}} = f'(r)\big[\mathbf{u_r}(\mathbf{u_r} \cdot \mathbf{p}_0) - \mathbf{p}_0\big] - \frac{f(r)}{r}\big[\mathbf{p}_0 + (\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r}\big] \tag{A.19}$$

$$= \frac{\mu_0\omega}{4\pi}e^{ikr}\left(\frac{ik^2}{r} - \frac{2k}{r^2} - \frac{2i}{r^3}\right)\big[\mathbf{u_r}(\mathbf{u_r} \cdot \mathbf{p}_0) - \mathbf{p}_0\big] - \frac{\mu_0\omega}{4\pi}e^{ikr}\big(k/r^2 + i/r^3\big)\big(\mathbf{p}_0 + (\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r}\big) \tag{A.20}$$

We introduce the prefactor

$$\tilde{\mathbf{E}}(\mathbf{r}) = \frac{ic^2}{\omega}\nabla \times \tilde{\mathbf{B}}.$$

and observe

$$\frac{ic^2}{\omega}\frac{\mu_0\omega}{4\pi} = \frac{i}{4\pi\varepsilon_0}.$$

Therefore,

$$\tilde{\mathbf{E}}(\mathbf{r}) = \frac{i}{4\pi\varepsilon_0}e^{ikr}\Big\{\big(\tfrac{ik^2}{r} - \tfrac{2k}{r^2} - \tfrac{2i}{r^3}\big)\big[\mathbf{u_r}(\mathbf{u_r} \cdot \mathbf{p}_0) - \mathbf{p}_0\big] - \big(\tfrac{k}{r^2} + \tfrac{i}{r^3}\big)\big[\mathbf{p}_0 + (\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r}\big]\Big\}.$$

After distribution of i and rearrangement of the dot products :

$$\tilde{\mathbf{E}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}e^{ikr}\left\{\left(\frac{-k^2}{r} - \frac{2ik}{r^2} + \frac{2}{r^3}\right)\big[(\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r} - \mathbf{p}_0\big] + \left(\frac{1}{r^3} - \frac{ik}{r^2}\right)\big[(\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r} + \mathbf{p}_0\big]\right\}.$$

We now regroup by the different terms in powers of r :

Complex amplitude of the electric field radiated by an oscillating dipole $\mathbf{p_0}$

$$\tilde{\mathbf{E}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} e^{ikr} \left\{ \frac{k^2}{r} [\mathbf{p_0} - (\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r}] + \frac{ik}{r^2} [\mathbf{p_0} - 3(\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r}] - \frac{1}{r^3} [\mathbf{p_0} - 3(\mathbf{p}_0 \cdot \mathbf{u_r})\mathbf{u_r}] \right\} \tag{A.21}$$

Dyadic form Using the definition of the dyadic product used above, we can express a more compact form, which is more frequent in textbook results :

Complex amplitude of a field radiated by an oscillating dipole $\mathbf{p_0}$

$$\tilde{\mathbf{E}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} e^{ikr} \left[k^2 \frac{\mathbf{I} - \mathbf{u_r}\mathbf{u_r}}{r} + (\mathbf{I} - 3\mathbf{u_r}\mathbf{u_r}) \left(\frac{ik}{r^2} - \frac{1}{r^3} \right) \right] \cdot \mathbf{p}_0 \tag{A.22}$$

Near field limit

Near-field limit

Upon inspection, we see that in the limit $kr \ll 1$, the phase term is now negligible and only the r^{-3} is preponderant :

$$\tilde{\mathbf{E}}(\mathbf{r}) = \frac{-1}{4\pi\varepsilon_0} \left[\frac{(\mathbf{I} - 3\mathbf{u_r}\mathbf{u_r})}{r^3} \right] \cdot \mathbf{p}_0 \quad \text{for} \quad kr \ll 1$$

Upon inspection, we see that equation (A.2) is strictly the same as (A.22) : in other words, in the regime $kr \ll 1$, the field radiated by an oscillating dipole is strictly equivalent to the field instantaneously generated by the dipole as if it was an electrostatic dipole.

This is another definition of the near-field : we define it as a spatial domain where no retardation effects are visible (hence the vanishing of the propagation exponential $\exp(ikr)$, and the consequence is that the field, even if we formally still work in harmonic regime, is given by electrostatics[a].

[a] This result can be retrieved using another approximation : considering an infinite velocity of light, so that there is no retardation effects upon propagation.

Concepts and key ideas

Introduction to near-field optics

- test