Benjamin Vest
benjamin.vest@institutoptique.fr
Correction au 1er octobre 2025

Master QLMN - Université Paris-Saclay

TD 5 - Non-equilibrium Statistical Physics
Boltzmann Equation - CORRECTION

1 Viscosity

$$f^{(0)} = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-m\frac{(\mathbf{v}-\mathbf{V})^2}{2k_B T}\right)$$

$$\Delta p_x = \int d^3\mathbf{v} p_x f(\mathbf{r},\mathbf{v})\mathbf{v}\cdot\mathbf{n} dt\Sigma$$

$$\Phi_{p_x,z} = \frac{\Delta p_x}{dt\Sigma} = \int d^3\mathbf{v} p_x f(\mathbf{r},\mathbf{v}) v_z$$

Boltzmann Equation :

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\vec{\nabla}_{\mathbf{r}} f(\mathbf{r},\mathbf{v}) + \frac{d\mathbf{v}}{dt}\cdot\vec{\nabla}_{\mathbf{v}} f(\mathbf{r},\mathbf{v}) = -\frac{f-f^{(0)}}{\tau}$$

$$v_z\frac{\partial f}{\partial z} = -\frac{f-f^{(0)}}{\tau} = -\frac{\phi f^{(0)}}{\tau}$$

Make the gradient of the inhomogeneous physical quantity appear :

$$\phi f^{(0)} = \tau\frac{\partial f}{\partial v_x}\frac{\partial V}{\partial z}$$

$\phi f^{(0)}$ is the correction to the velocity distribution and is responsible for the flux existing our of equilibrium.

$$\begin{aligned}\Phi_{p_x,z} &= \int d^3\mathbf{v} p_x f(\mathbf{r},\mathbf{v}) v_z \\ &= -\mu\frac{\partial V}{\partial z} \\ &= \iiint d^3\mathbf{v} p_x v_z[\tau v_z]\frac{\partial f^{(0)}}{\partial v_x}\frac{\partial V}{\partial z}\end{aligned}$$

We write :

$$f(\mathbf{r}, \mathbf{v}) = nP(v_x)P(v_y)P(v_z)$$

so that :

$$\begin{aligned}
\Phi_{p_x,z} &= \iiint d^3\mathbf{v} p_x v_z [\tau v_z] \frac{\partial f^{(0)}}{\partial v_x} \frac{\partial V}{\partial z} \\
&= mn\tau \underbrace{\left[\int dv_x v_x \frac{dP(v_x)}{dv_x}\right]}_{=[v_x P(v_x)]_{-\infty}^{+\infty} - \int P(v_x) dv_x = -1} \underbrace{\left[\int dv_y P(v_y)\right]}_{=1} \underbrace{\left[\int dv_z v_z^2 P(v_z)\right]}_{=<v_z^2>=\frac{k_B T}{m}} \frac{\partial V}{\partial z} \\
&= -n\tau k_B T \frac{\partial V}{\partial z}
\end{aligned}$$

$$\mu = n\tau k_B T$$

1.1 Recalling results from the previous tutorial

Viscosity is a measure of internal friction between adjacent laers of a fluid. This friction can be seen as a transport of momentum between layers by particles in the fluid. In this exercise, we consider a laminar flow $V_x(z)\mathbf{e_x}$. (the liquid flows in the x-direction, but layers at different$z$ can have a different velocity). We use elements of the kinetic theory to solve the problem.

Viscosity μ is given by $F_{xz} = -\mu \frac{\partial V_x}{\partial z}$. Let us consider of small volume, defined by a cross-section Σ separating two layers and a lateral extension zd. The net momentum transfer in this layer is given by contributions from upper and lower layers closer than the free mean path, and from particles with appropriate velocities. We consider that particles have a random thermal velocity v, same in all directions, that adds up to the collective laminar flow velocity V_x.

$$\begin{aligned}
\frac{dp_{syst,x}}{dt} &= \Sigma \frac{1}{6} \left(nv_z[mV_x(z-l)] - nv_z[mV_x(z+l)] \right) \\
&= -\Sigma \frac{n}{6} vml2 \frac{\partial V_x}{\partial z}
\end{aligned}$$

The shear force is a force per unit surface. We divide the previous momentum transfer by Σ and use the other expressions of the mean free path $l = v\tau$ and root mean square velocity $v^2 = \frac{3kT}{m}$ to get :

$$F_{xz} = -nkT\tau \frac{\partial V_x}{\partial z}$$

and we identify directly the viscosity : $\mu = nkT\tau$.

2 Diffusion current in a p-n junction

$$\mathbf{j_n} = -D\vec{\nabla}_\mathbf{r} n(\mathbf{r}) = \int (-e) \underbrace{f(\mathbf{r}, \mathbf{v})}_{=f^{(0)} + \phi f^{(0)}} \mathbf{v} d^3\mathbf{v}$$

$$n(\mathbf{r}) = \int f(\mathbf{r}, \mathbf{v}) d^3\mathbf{v}$$

$$f(\mathbf{r}, \mathbf{v}) = n(\mathbf{r}) P(v_x) P(v_y) P(v_z)$$

Boltzmann equation :

$$\mathbf{v} \cdot \vec{\nabla}_{\mathbf{r}} f(\mathbf{r}, \mathbf{v}) = -\frac{\phi f^{(0)}}{\tau}$$

$$\begin{aligned} \mathbf{j_n} &= -D\vec{\nabla}_{\mathbf{r}} n(\mathbf{r}) \\ &= \iiint e\tau \left(\mathbf{v} \cdot \vec{\nabla}_{\mathbf{r}} f(\mathbf{r}, \mathbf{v}) \right) \mathbf{v} d^3\mathbf{v} \\ &= \iiint e\tau \left(\mathbf{v} \cdot \vec{\nabla}_{\mathbf{r}} n(\mathbf{r}) \right) P(\mathbf{v}) \mathbf{v} d^3\mathbf{v} \end{aligned}$$

For the x component :

$$\begin{aligned} j_{n,x} &= \left(\iiint e\tau \left(\mathbf{v} \cdot \vec{\nabla}_{\mathbf{r}} n(\mathbf{r}) \right) P(\mathbf{v}) \mathbf{v} d^3\mathbf{v} \right) \cdot \mathbf{e_x} \\ &= e\tau \iiint dv_x dv_y dv_z \left[v_x^2 \frac{\partial n}{\partial x} + v_x v_y \frac{\partial n}{\partial y} + v_x v_z \frac{\partial n}{\partial z} \right] P(v_x) P(v_y) P(v_z) \\ &= e\tau \left(\underbrace{< v_x^2 >}_{\frac{k_B T}{m}} \frac{\partial n}{\partial x} + \underbrace{< v_x >< v_y > \frac{\partial n}{\partial y}}_{=0} + \underbrace{< v_x >< v_z > \frac{\partial n}{\partial z}}_{=0} \right) \end{aligned}$$

so that :

$$\mathbf{j_n} = -D\vec{\nabla}_{\mathbf{r}} n(\mathbf{r}) = \frac{e\tau k_B T}{m} \vec{\nabla}_{\mathbf{r}} n(\mathbf{r})$$

this is valid if Maxwell Boltzmann distribution can be used, i.e. if we are in the classical approximation.